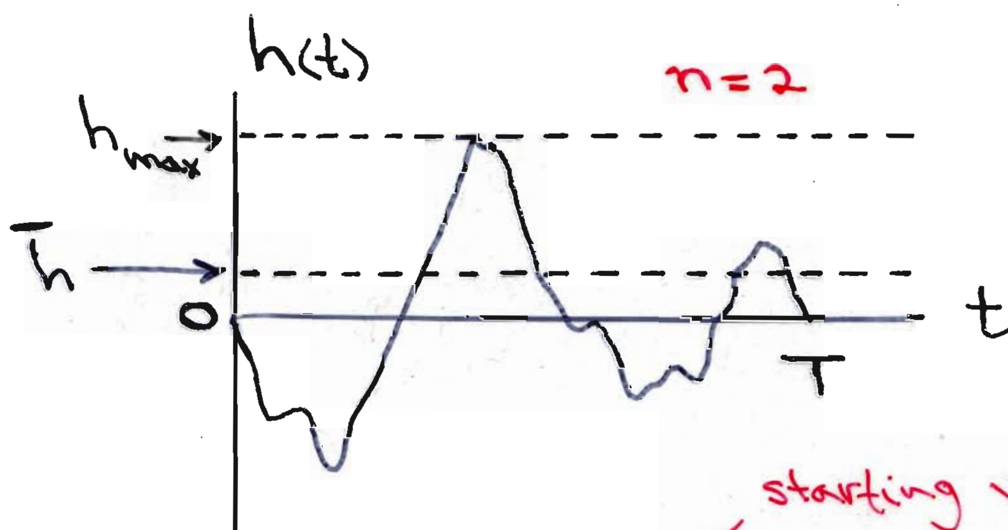
Extreme Statistics of Random acceleration and Related Processes

$$\frac{d^n h}{dt^n} = \xi(t) \leftarrow \text{Gaussian white noise}$$

$n=0$	$h(t) = \xi(t)$	iid variable
$n=1$	random walk	
$n=2$	random acceleration	



starting value

1. Distribution of $m = h_{max} - h(0)$

TWB, Györgyi, Moloney, Rácz PRE 2007

Distribution of $h_{max} - \bar{h}$

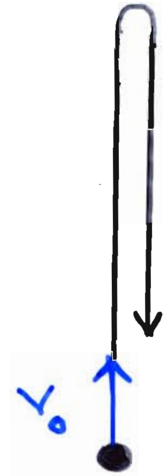
Majumdar, Comtet JSP 2005

Györgyi, Moloney, Ozogány, Rácz PRE 2007

2. Distribution of maximum height for

random

$$\frac{d^2h}{dt^2} = -g + \xi(t)$$



TWB JSP 2008

3. Extreme statistics and confined polymers

$$\frac{d^n h}{dt^n} = \xi(t)$$

Distribution of $m = h_{max} - h(0)$ periodic boundary conditions

Exact results for $n = 0, 1, 2, \infty$. Numerical simulations for other n.

$$\text{probability} \propto e^{-K\int_0^T dt \left(\frac{d^n h}{dt^n}\right)^2} = e^{-KT\sum_\omega \omega^{2n} |h_\omega|^2}$$

$$h(t) = \sum_\omega h_\omega e^{i\omega t}, \qquad 0 < t < T$$

$$\omega = \frac{2\pi m}{T} \qquad m = \frac{N}{2}, \frac{N}{2} - 1, \ldots, -\frac{N}{2} + 1$$

$n=0$ $\quad h_0, h_1, h_2, \ldots, h_N \leftarrow$ iid variables Gaussian pdf $\rho(h)$

what is distribution of $m = h_{max} - h_0$?

$\mathcal{F}_N(m) \equiv$ prob. that $h_1 - h_0 < m,\ h_2 - h_0 < m, \ldots$ $h_N - h_0 < m$

$$\mathcal{F}_N(m) = \int_{-\infty}^{\infty} dh_0\, \rho(h_0) \underbrace{\int_{-\infty}^{m+h_0} dh_1\, \rho(h_1) \cdots \int_{-\infty}^{m+h_0} dh_N\, \rho(h_N)}_{\exp(-e^{-z})}$$

Fisher Tippett Gumbel

$z = a_N^{-1}(m + h_0 - b_N)$

$a_N \sim (\ln N)^{-1}, \quad b_N \sim (\ln N)^{1/2}$

$$P_N(m) = \frac{\partial \mathcal{F}_N(m)}{\partial m} = \rho(b_N - m)$$

Distribution of $m = h_{max} - h_0$ is Gaussian.

Distribution of h_{max} and $h_{max} - \bar{h}$ are of FTG type.

Results for other $\rho(h)$

$\rho(h) \sim \exp(-h^{\delta})$, $h \to \infty$, $\delta > 0$

Distributions of h_{max} and $h_{max} - \bar{h}$ have FTG form.

Distribution of $m = h_{max} - h_0$

$\delta > 1$ $\quad a_N \to 0$ $\qquad P_N(m) = \rho(b_N - m)$

$\delta < 1$ $\quad a_N \to \infty$ $\qquad$ FTG

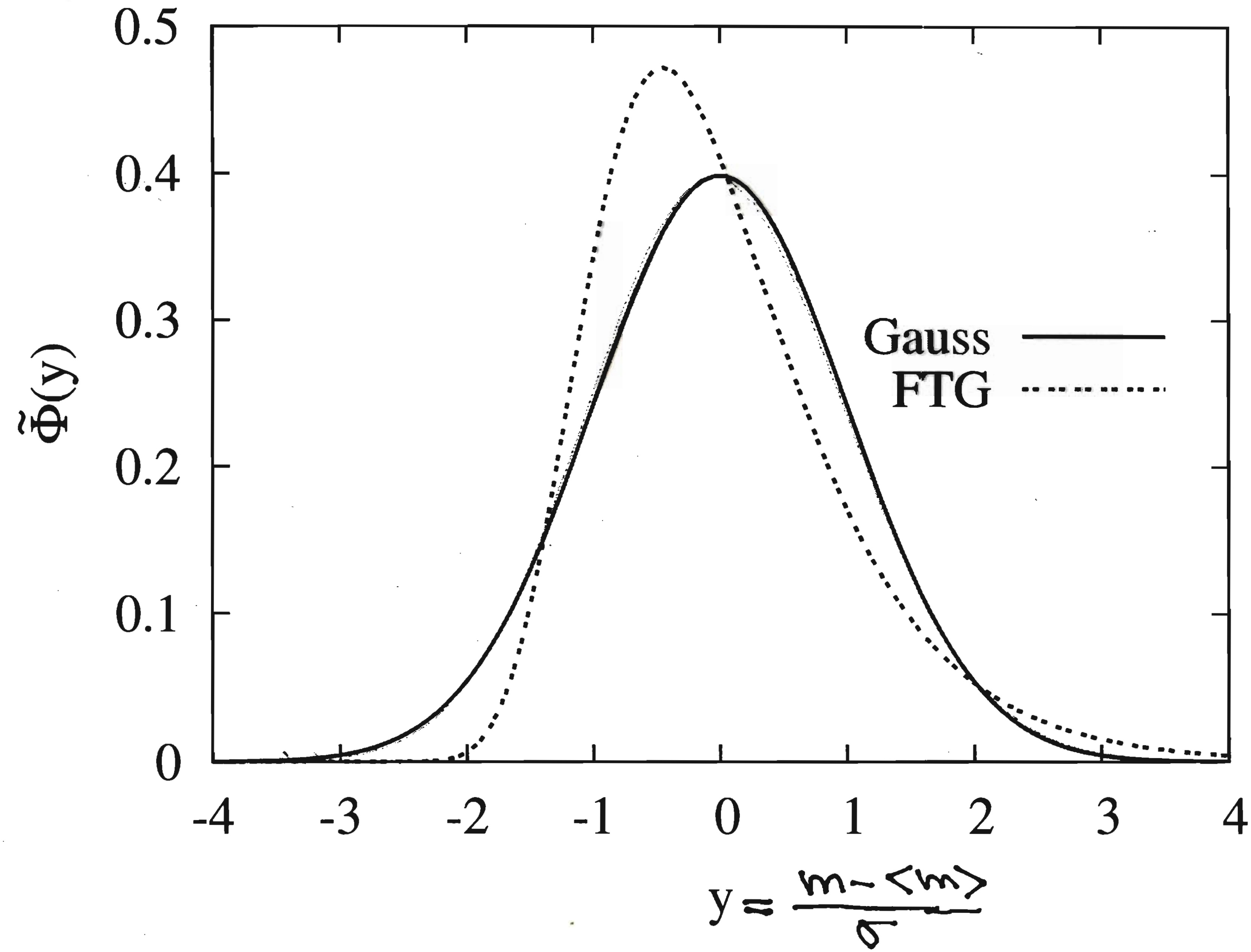
$\delta = 1$ $\quad a_N \to a$ $\qquad R(m - b_N)$

$$R(x) = \int_{-\infty}^{\infty} dz\, \rho(az - x) \exp\left(-z - e^{-z}\right)$$

$\rho(h)$ with other large h behavior considered.

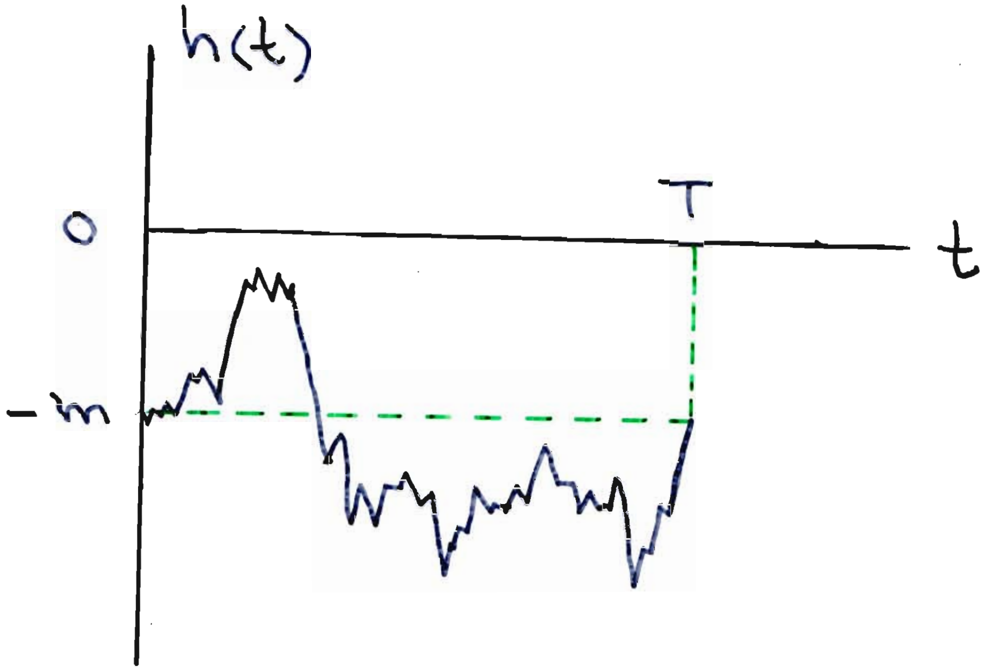
$m = h_{max} - h_0$

$P_N(m)$ essentially same as the density of near extreme states considered by Sabhapandit and Majumdar, PRL 2007.


$\tilde{\Phi}(y)$
0.5
0.4
0.3
0.2
0.1
0
-4
-3
-2
-1
0
1
2
3
4
Gauss
FTG
$y = \frac{m - \langle m \rangle}{\sigma}$

n = 1

random walk



$\mathcal{F}(m,T)$ = prob. that $h_{max} - h(0)$ does not exceed m in time T

$$= \frac{Z_{\text{half space}}(-m,-m,T)}{Z_{\text{bulk}}(-m,-m,T)}$$

$$Z(h,h_0,t) = \frac{1}{\sqrt{4\pi t}}\left[e^{-(h-h_0)^2/4t} - e^{-(h+h_0)^2/4t}\right]$$

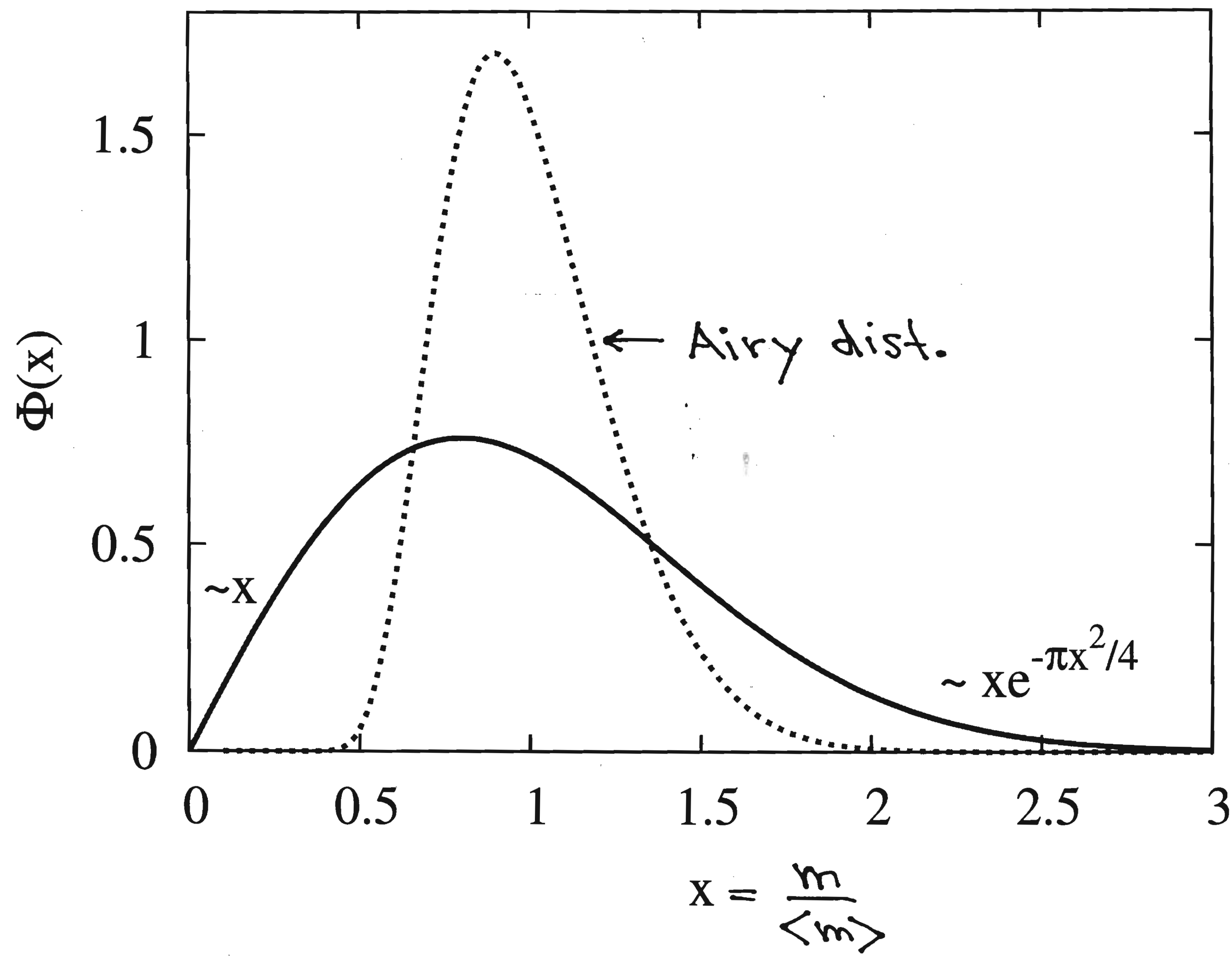
propagator for half space $h<0$, absorbing boundary

$$P(m,T) = \frac{\partial \mathcal{F}(m,T)}{\partial m} = \frac{2m}{T} e^{-m^2/T}$$

dist. of $m = h_{max} - h(0)$

Dist. of $h_{max} - \overline{h}$ is airy distribution

Majumdar, Comtet JSP 2005


Φ(x)
1.5
1
0.5
0
0
0.5
1
1.5
2
2.5
3
~x
← Airy dist.
~ xe^(-πx²/4)
x = m/⟨m⟩

$n = 2$ $\quad \frac{d^2h}{dt^2} = \xi(t)$ random acceleration

$$\mathcal{F}(m,T) = \frac{\int_{-\infty}^{\infty} dv\, Z_{\text{half space}}(m,v;m,v,T)}{\int_{-\infty}^{\infty} dv\, Z_{\text{bulk}}(\text{same})}$$

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial h} - \frac{\partial^2}{\partial v^2}\right) Z_{\text{half space}}(h,v;h_0,v_0;t) = 0$$

absorbing boundary condition $\quad Z_{\text{half space}}(0,v;h_0,v_0;t) \leftarrow$ vanishes for $v>0$ but not $v<0$

separable solution $\quad e^{-Fh+st}\, Ai(-F^{1/3}v + F^{2/3}s)$

$$\tilde{Z}_{\text{half space}}(h,v;h_0,v_0;s) = \tilde{Z}_{\text{bulk}}(\text{same})$$

$$- \frac{1}{2\pi}\int_0^\infty \frac{dF}{F^{1/6}} \int_0^\infty \frac{dG}{G^{1/6}} \frac{e^{-Fh-Gh_0-\frac{2}{3}s^{3/2}(F^{-1}+G^{-1})}}{F+G}$$

$$Ai(-F^{1/3}v + F^{2/3}s)\, Ai(G^{1/3}v_0 + G^{2/3}s)$$

McKean 1963

Marshall and Watson 1985, TWB 1993 $\quad \ddot{h} = -\xi(t) - \lambda\dot{h}$

$$P(m,T) = \frac{\partial \mathcal{F}(m,T)}{\partial m}$$

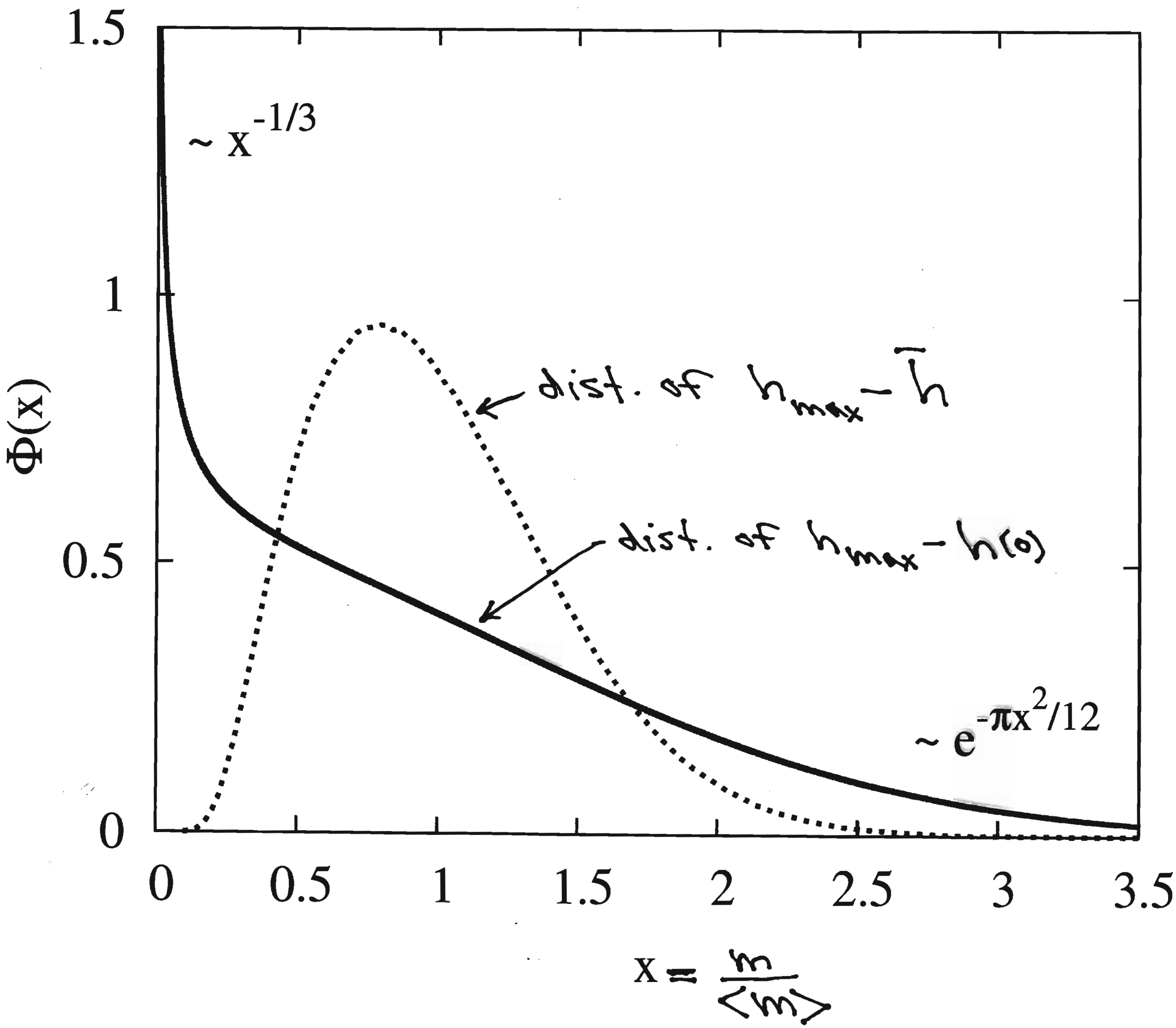
$$P(m,T) = c x^{-1/3} e^{-\pi x^2/12} U\left(-\tfrac{1}{6}, \tfrac{2}{3}, \tfrac{\pi x^2}{12}\right)$$

$$x = \frac{m}{\langle m \rangle} \propto \frac{m}{T^{3/2}}$$

Distribution $P(\tau, T)$ of the time τ to go from $h(0)$ to h_{max}

$$P(\tau,T) = c\,\delta(\tau - T) + \frac{1-c}{\pi\sqrt{2}}\,\tau^{-3/4}(T-\tau)^{-1/4}, \quad c = 1 - \sqrt{\tfrac{3}{8}}$$

Majumdar, Rosso, Zoia JPA 2010



1.5
1
0.5
0
$\Phi(x)$
$\sim x^{-1/3}$
dist. of $h_{max} - \bar{h}$
dist. of $h_{max} - h(0)$
$\sim e^{-\pi x^2/12}$
0
0.5
1
1.5
2
2.5
3
3.5
$x = \frac{m}{\langle m \rangle}$

$n = \infty$

$$\text{probability} \propto e^{-KT\sum_{\omega>0} \omega^{2n} |h_\omega|^2}$$

Only modes with smallest ω contribute

$$h(t) - h(0) = a e^{i\phi}\left(e^{2\pi i t/T} - 1\right) + \text{c.c.}$$

$$h_{max} = 4a \sin^2\frac{\phi}{2}$$

$$P(m) \propto x^{-1/2} e^{-\pi x^2/16}\, U\!\left(-\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{\pi x^2}{16}\right)$$

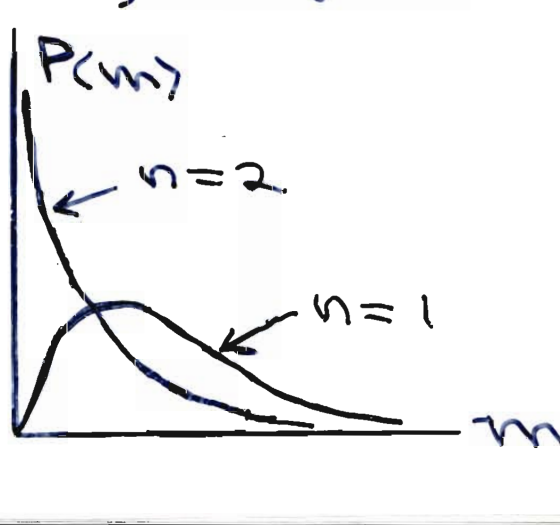
$$x = \frac{m}{\langle m \rangle}$$

Dist. of $m = h_{max} - h \propto x e^{-\pi x^2/4}$

Main features of extreme distribution

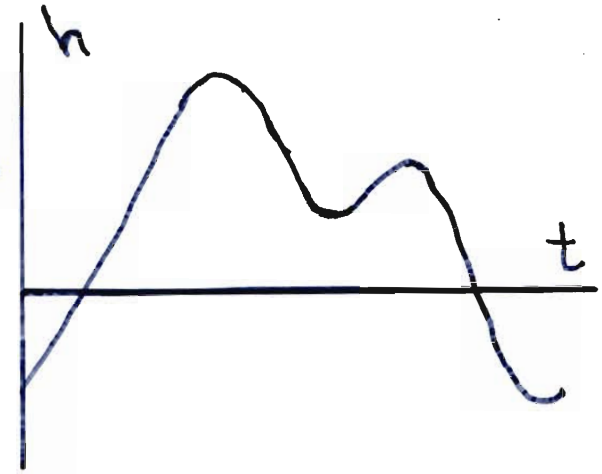
$$P(m,T) \sim \begin{cases} m^{\gamma(n)}, & \text{small } m \\ e^{-\alpha m^2/T^{2n-1}}, & \text{large } m \end{cases}$$

$\gamma(n)$ decreases as n increases, changes sign between $n=1$ and 2



Every point on periodic curve can be interpreted as starting point.

For large n maximum is parabolic.



$$P(m) \propto \int dt\, \delta(m - \alpha (t-t^*)^2) \propto m^{-1/2}$$

agrees with exact result for $n=\infty$ and should hold at least down to $n=3$ but not $n=2$.

$$\frac{d^n h}{dt^n} = \xi(t)$$

$t^{n-1/2}$

$$h(t) - h(0) = \int_0^t dt' \frac{(t-t')^{n-1}}{(n-1)!} \xi(t') + \begin{cases} 0 & \leftarrow n=1 \\ v_0 t & \leftarrow n=2 \\ v_0 t + \frac{1}{2} a_0 t^2 & \leftarrow n=3 \end{cases}$$

For smaller n

$$P(m) \propto \int dt\, \delta(m - \alpha (t-t^*)^{n-1/2}) \propto m^{(\frac{3}{2}-n)/(n-\frac{1}{2})}$$

$$\gamma(n) = \begin{cases} -\frac{1}{2}, & \frac{5}{2} < n < \infty \\ (\frac{3}{2}-n)/(n-\frac{1}{2}), & \frac{1}{2} < n < \frac{5}{2} \\ \infty, & 0 < n < \frac{1}{2} \end{cases}$$

agrees with exact results for $n = 0, \frac{1}{2}, 1, 2, \infty$ and simulations.

For $n=0$ $\quad P_N(m) \propto e^{-\lambda(m-a\ln N)^2}$

all derivatives vanish at $m=0$, just as for m^γ, $\gamma=\infty$.

$n=0$ results hold up to $n=\frac{1}{2}$. Berman, AMS 1964

close connection between $P(m)$ and density of states near maximum studied by Sabhapandit and Majumdar, PRL 2007.

Particle Subject to Gravity and a Random Force

$$\frac{d^2h}{dt^2} = -g + \xi(t)$$

$$\langle \xi(t)\xi(t') \rangle = 2\Lambda\delta(t-t')$$

scale h and t so $g=1$, $\Lambda=1$

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial h} - g\frac{\partial}{\partial v} - \frac{\partial^2}{\partial v^2}\right) Z_g(h,v;h_0,v_0;t) = 0$$

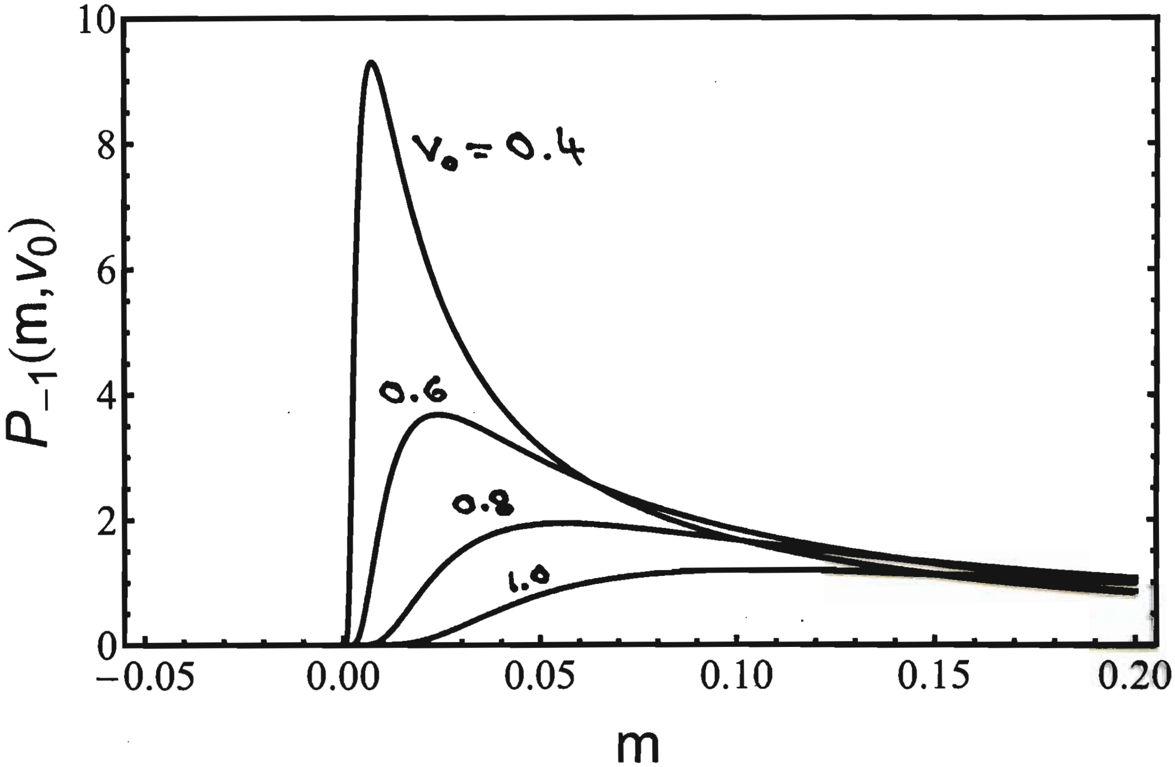
$$Z_g(h,v;h_0,v_0;t) = e^{-\frac{1}{2}(v-v_0)-\frac{1}{4}t}\, Z_0(\text{same})$$

$$\mathcal{F}(m,v_0) = \text{probability that } \max_{0<t<\infty} h(t) < m$$

$$P(m,v_0) = \frac{\partial \mathcal{F}(m,v_0)}{\partial m}$$

$$P(m,v_0) = \Theta(-v_0)\,\mathrm{erf}\left(\sqrt{\frac{3}{2}|v_0|}\right)\delta(m)$$

$$+\,\Theta(m)\frac{e^{v_0/2}}{\sqrt{\pi}}\int_0^\infty dF\, F^{-1/6}\exp\left(-\frac{1}{12F} - Fm\right)\mathrm{Ai}\left(-F^{1/3}v_0 + \frac{1}{4}F^{-2/3}\right)$$



For large v_0

$$\langle m \rangle = \frac{v_0^2}{2g} + \frac{\Lambda}{g^2} v_0$$

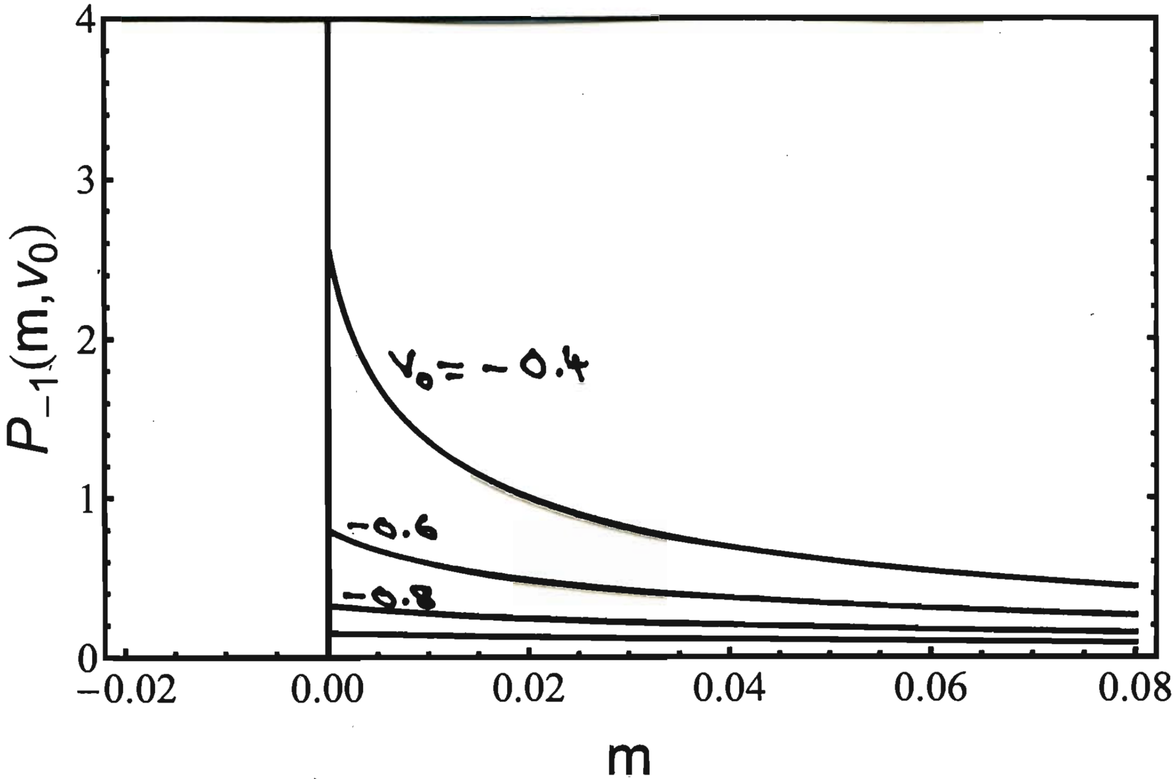
$$\frac{1}{2} v_0^2 = gh$$

$$\sigma = \sqrt{\frac{2}{3}}\, \Lambda^{1/2} \left(\frac{v_0}{g}\right)^{3/2}$$

$$\Delta h \sim \Lambda^{1/2} t^{3/2}$$

In limit $\Lambda \to 0$

$$P_g(m, v_0) \longrightarrow \delta\left(m - \frac{v_0^2}{2g}\right)$$



For $v_0 < 0$, probability that particle never returns to initial height is

$$\operatorname{erf}\left(\sqrt{\frac{3g}{2\Lambda}\,|v_0|}\right).$$

Extreme statistics and confined polymers

$$\mathcal{F}(R) = \frac{Z_R}{Z_\infty} = e^{-\Delta F / kT}$$

prob. that any two points of polymer are separated by distance less than $2R$

sphere of radius R

ΔF = free energy of confinement ← work to squeeze polymer into sphere

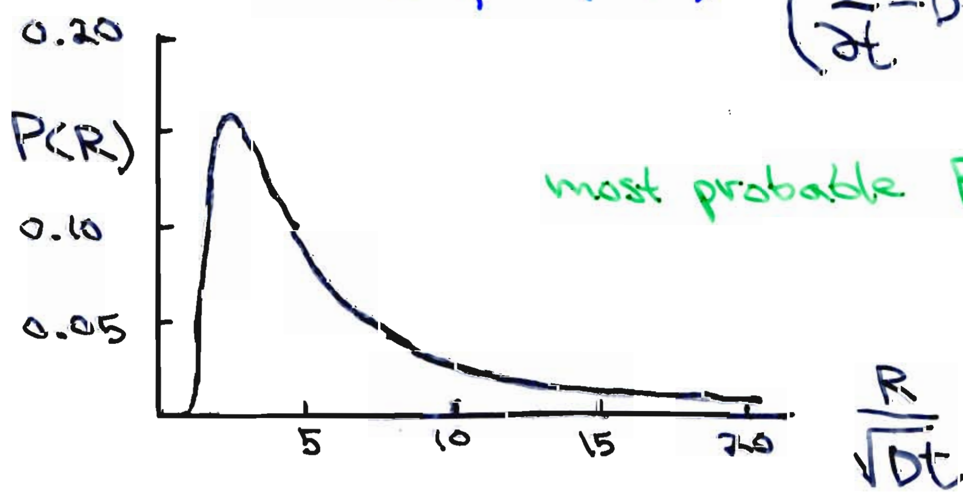
$$\Delta F = -kT \ln \mathcal{F}(R)$$

extreme dist. $\mathcal{F}'(R)$

$$\text{pressure} = -\frac{\partial \Delta F}{\partial V} = \frac{kT}{4\pi R^2} \frac{P(R)}{\mathcal{F}(R)}$$

Results for random walk (ideal polymer)

$$\left(\frac{\partial}{\partial t} - D\nabla^2\right) Q_R(\vec{r}, \vec{r}_0, t) = 0$$



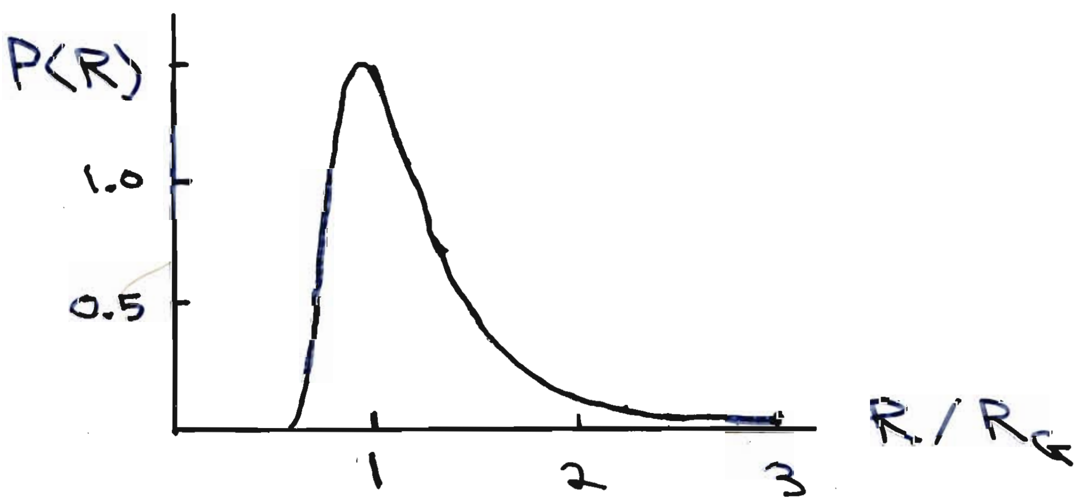
most probable $R = 2.6\sqrt{Dt}$

flexible, nonideal polymer $\quad R_G = aN^{\nu}, \quad \nu = 0.59$

$$\frac{\Delta F}{kT} \sim \left(\frac{R_G}{R}\right)^{3/(3\nu-1)} \quad \nwarrow 3.9$$

Grosberg & Khokhlov 1994
Coccinato & Luijten 2005

$$\mathcal{F}(R) = e^{-\Delta F/kT}, \quad P(R) = \mathcal{F}'(R)$$

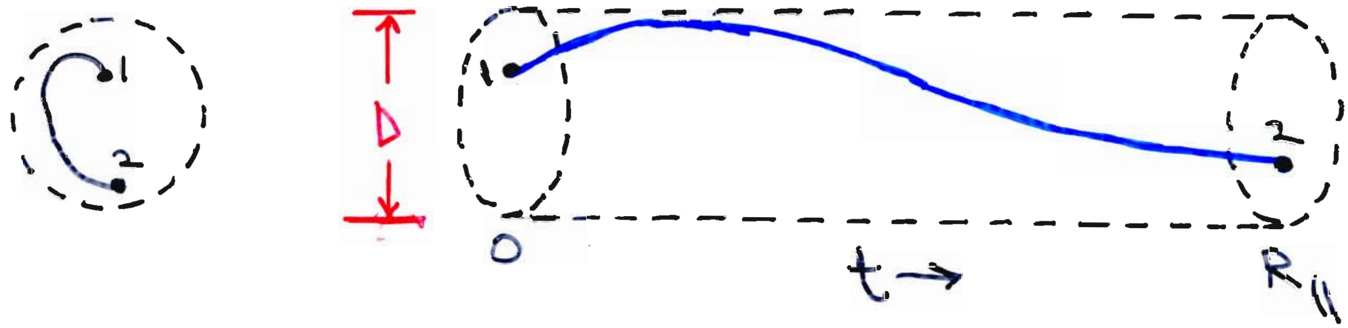


Most probable $R = 0.94\, R_G$

Long semiflexible polymer in narrow tube

$D \ll l_p \ll L$ ← contour length

↑ diameter ↖ persistence length



$$Z_D = \int \mathcal{D}^2 r \, e^{-\mathcal{H}/kT}$$

$\vec{r} = (x, y)$

$$\mathcal{H}/kT = \frac{1}{2} l_p \int_0^L ds \left(\frac{d\hat{t}}{ds}\right)^2 \longrightarrow \frac{1}{2} l_p \int_0^{R_\parallel} dt \left(\frac{d^2\vec{r}}{dt^2}\right)^2 \quad \leftarrow \frac{d^2\vec{r}}{dt^2} = \vec{\xi}(t)$$

$$\mathcal{F}(D) = \frac{Z_D}{Z_\infty} = e^{-\Delta F/kT}$$

↑ prob. that randomly accelerated particle in D=2 has not left circular domain within time t

$$\frac{\Delta F}{kT} = A \, l_p^{-1/3} D^{-2/3} R_\parallel \, , \quad A = 2.36$$

$\vec{r} = D \vec{r}\,'$

$t = l_p^{1/3} D^{2/3} t'$

TWB, Bicout, Yang, Gompper

$$\mathcal{F}(D) = e^{-A l_p^{-1/3} D^{-2/3} R_\parallel}$$